Group homology and ideal fundamental cycles

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Abstract

We prove that the (homological version of the) generalized Goncharov invariant of locally symmetric spaces determines their generalized Neumann-Yang invariant.

1 Introduction

Let M be an odd-dimensional locally symmetric space of noncompact type, with $vol(M) < \infty$, which is either compact or of rank one. We compare two invariants: the generalized Goncharov invariant and the generalized Neumann-Yang invariant, and we show that the former determines the latter.

In [7], Walter D. Neumann and Jun Yang constructed an invariant of finite-volume hyperbolic 3-manifolds which lives in the Bloch group $\mathcal{B}(\mathbf{C})$, and from which volume and Chern-Simons invariant can be recaptured. On the other hand, Alexander Goncharov [5] constructed an invariant of odd-dimensional hyperbolic manifolds and representations of their fundamental groups. (In particular he considered the half-spinor representations.) This invariant lives in $K_*(\overline{\mathbf{Q}}) \otimes \mathbf{Q}$ and the volume can be recaptured by means of the Borel regulator. In [6], we generalized Goncharov's construction to finite-volume locally symmetric spaces of noncompact type $\Gamma \backslash G/K$ (either closed or locally rank one symmetric) and representations $\rho : G \to GL(N, \mathbf{C})$. We constructed an invariant $\overline{\gamma}(M) \in H_d(GL(N, \overline{\mathbf{Q}}); \mathbf{Q})$ and a projection $\pi : H_d(GL(\overline{\mathbf{Q}}); \mathbf{Q}) \to K_d(\overline{\mathbf{Q}}) \otimes \mathbf{Q}$ yielding an invariant $\gamma(M) := \pi(\overline{\gamma}(M)) \in K_d(\overline{\mathbf{Q}}) \otimes \mathbf{Q}$, and we analyzed in which cases the obtained invariant $\gamma(M)$ is nontrivial resp. trivial. (The nontriviality of $\gamma(M)$ turned out to depend only on the representation $\rho : G \to GL(N, \mathbf{C})$. Again, if $\gamma(M) \neq 0$, then $vol(M)$ can be recaptured from $\gamma(M)$ by application of the Borel regulator.)

Here we show that the natural evaluation map ev from $H_*(GL(N, \mathbf{C}))$ to the generalized pre-Bloch groups maps $\overline{\gamma}(M)$ to the natural generalization of the Neumann-Yang invariant $\beta(M)$. The latter invariant is defined using ideal fundamental cycles.

Theorem 1. *Let $\overline{M}$ be a compact, orientable, connected manifold with boundary such that $M := \overline{M} - \partial \overline{M} = \Gamma \backslash G/K$ is a finite-volume, locally rank one symmetric space of noncompact type. Let $\rho : G \to GL(N, \mathbb{C})$ be a representation. Then*

$$ev_*\left(\overline{\gamma}\left(\overline{M}\right)\right) = \beta_\rho\left(\overline{M}\right).$$

For $N = 2$, we can derive the following consequence:

Corollary 2. *If M^3 is hyperbolic of finite volume, then Suslin's homomorphism*

$$K_3(\mathbf{C}) \otimes \mathbf{Q} \to \mathcal{B}(\mathbf{C}) \otimes \mathbf{Q}$$

maps the Goncharov invariant $\gamma\left(M\right)$ to the Neumann-Yang invariant $\beta\left(M\right)\otimes 1$.

Although Corollary 2 seems to be well-known to the experts, at least for the case of closed manifolds, we could not locate a reference.

This paper makes essential use of the definition of the Goncharov invariant for cusped manifolds ([6, Section 4] and as such is a continuation and extension of [6]. We decided to publish it separately because [6] already has a considerable length. The paper is organized as follows: Section 2 states the definitions of the Goncharov invariant and the (generalized) Neumann-Yang invariant. Section 3 provides a chain map $\hat{\Psi}$ from relative chains to ideal chains which, in particular, sends relative fundamental cycles to ideal fundamental cycles. This is used in Section 4 to prove Theorem 1.

2   Definitions

2.1   Goncharov invariant

Group homology. For a group G, its classifying space (with respect to the discrete topology on G) is the simplicial set BG defined as follows:
   - the set of k-simplices of BG is

$$S_k^{simp}\left(BG\right)=\left\{\left(g_1,\ldots,g_k\right):g_1,\ldots,g_k\in G\right\},$$

   - the boundary operators are defined by
   $\partial_0\left(g_1,\ldots,g_k\right)=\left(g_2,\ldots,g_k\right),$
   $\partial_i\left(g_1,\ldots,g_k\right)=\left(g_1,\ldots,g_ig_{i+1},\ldots,g_k\right)$ for $i=1,\ldots,k-1,$
   $\partial_k\left(g_1,\ldots,g_k\right)=\left(g_1,\ldots,g_{k-1}\right).$
   - the degeneracy maps are defined by $s_j\left(g_1,\ldots,g_k\right)=\left(g_1,\ldots,g_j,1,g_{j+1},\ldots,g_k\right).$
$C_k\left(BG\right)$ is the free abelian group freely generated by $S_k\left(BG\right)$. Let $\partial:C_k\left(BG\right)\rightarrow C_{k-1}\left(BG\right)$ be the linear extension of $\partial=\sum_{i=0}^{k}\left(-1\right)^i\partial_i$. For an abelian group R, the group homology $H_*\left(G;R\right)=H_*^{simp}\left(BG;R\right)$ is the homology of $\left(C_*\left(BG\right)\otimes_{\mathbf{Z}}R,\partial\otimes 1\right)$.

If $\Gamma=\pi_1 M$ for an aspherical space M, then $H_*^{simp}\left(B\Gamma;R\right)=H_*\left(M;R\right)$. In particular, this is the case if M is a locally symmetric space of noncompact type.

Homological Goncharov invariant of closed manifolds. Let $M^d=\Gamma\backslash G/K$ be a closed, orientable, connected, locally symmetric space of noncompact type, and $\rho:G\rightarrow SL\left(N,\mathbf{C}\right)$ a representation[1] . Let $[M]$ be a generator of $H_d\left(M;\mathbf{Z}\right)\simeq\mathbf{Z}$. Then the homological Goncharov invariant of $\left(M,\rho\right)$ is defined as

$$\overline{\gamma}\left(M\right):=B\left(\rho j\right)_d EM_d^{-1}\left[M\right]\in H_d^{simp}\left(BSL\left(N,\mathbf{C}\right);\mathbf{Z}\right),$$

with $j:\Gamma\rightarrow G$ the inclusion, $B\left(\rho j\right):B\Gamma\rightarrow BSL\left(N,\mathbf{C}\right)$ induced by ρj and $EM_*^{-1}:H_*^{simp}\left(B\Gamma;\mathbf{Z}\right)\rightarrow H_*\left(M;\mathbf{Z}\right)$ the Eilenberg-MacLane isomorphism (cf. [6], Section 2.1).

(In [6], $\overline{\gamma}\left(M\right)$ was considered as an element of $H_d\left(BGL\left(\mathbf{C}\right);\mathbf{Q}\right)$. Moreover, in Section 2.5. of [6] we constructed a projection $H_d\left(BGL\left(\mathbf{C}\right);\mathbf{Q}\right)\rightarrow PH_d\left(BGL\left(\mathbf{C}\right);\mathbf{Q}\right)\simeq$

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[1]Since G is semisimple, any representation $\rho:G\rightarrow GL\left(N;\mathbf{C}\right)$ has image in $SL\left(N,\mathbf{C}\right).$

$K_d(\mathbf{C}) \otimes \mathbf{Q}$ and defined the **K-theoretic Goncharov invariant** $\gamma(M) \in K_d(\mathbf{C}) \otimes \mathbf{Q}$ as the image of $\overline{\gamma}(M)$ under this projection.

In this paper we will, except for Corollary 2, only use the homological invariant $\overline{\gamma}(M)$. The same remark applies to cusped manifolds below.)

Homological Goncharov invariant of cusped manifolds. We recall some notation from [6, Section 4.2]: For a continuous mapping : $A_1 \dot\cup \ldots \dot\cup A_s \to X$ we define the **disjoint cone**

$$DCone\left(\cup_{i=1}^s A_i \to X\right)$$

to be the pushout of the diagram

$$
\begin{array}{ccc}
A_1 \dot\cup \ldots \dot\cup A_s & \xrightarrow{\quad i \quad} & X \\
\downarrow & & \downarrow \\
Cone(A_1) \dot\cup \ldots \dot\cup Cone(A_s) & \longrightarrow & DCone\left(\cup_{i=1}^s A_i \to X\right)
\end{array}
$$

If X is a CW-complex and $A_1, \ldots, A_s$ are disjoint sub-CW-complexes, then clearly

$$H_*\left(DCone\left(\cup_{i=1}^s A_i \to X\right)\right) \cong H_*\left(Cone\left(\cup_{i=1}^s A_i \to X\right)\right) = H_*\left(X, \cup_{i=1}^s A_i\right)$$

in degrees $* \geq 2$. Similarly, for a simplicial mapping of simplicial sets $B_1 \dot\cup \ldots \dot\cup B_s \to Y$ we defined $DCone\left(\cup_{i=1}^s B_i \to Y\right)$ to be the quasi-simplicial set whose q-simplices are either q-simplices in Y or cones over $q-1$-simplices in some B_i, with the natural boundary operator.

Proposition 1. *Let $\overline{M}$ be a compact, orientable, connected d-manifold with boundary components $\partial_1\overline{M}, \ldots, \partial_s\overline{M}$ such that $M := \overline{M} - \partial\overline{M} = \Gamma\backslash G/K$ is a finite-volume locally rank one symmetric space of noncompact type. Let $\rho : G \to SL(N, \mathbf{C})$ be a representation such that $\Gamma_i' := \rho(\Gamma_i)$ is unipotent for $i = 1, \ldots, s$, where $\Gamma_i \subset \Gamma$ is the subgroup of $\pi_1(M, x)$ corresponding to[2] $\pi_1(\partial_i M, x_i)$. Denote*

$$\left[\overline{M}, \partial\overline{M}\right] \in H_d\left(DCone\left(\cup_{i=1}^s \partial_i\overline{M} \to \overline{M}\right); \mathbb{Z}\right)$$

the fundamental class of $\overline{M}$. Then

$$B(\rho j)_d EM_d^{-1}\left[\overline{M}, \partial\overline{M}\right] \in H_d\left(DCone\left(\cup_{i=1}^s B\Gamma_i' \to BSL(N, \mathbf{C})\right); \mathbb{Z}\right)$$

has a preimage

$$\overline{\gamma}\left(\overline{M}\right) \in H_d\left(BSL(N, \mathbf{C}); \mathbb{Z}\right).$$

This is proved in [6, Proposition 2].

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[2]Our assumptions imply that $\pi_1\partial_i M$ injects into $\pi_1 M$, cf. [6], Observation 1. In particular, if we fix $x_0 \in M$ and $x_i \in \partial_i M$ for $i = 1, \ldots, s$, then we obtain (using some path from x_0 to x_i) **isomorphisms of $\pi_1(\partial_i M, x_i)$ with subgroups Γ_i of** $\Gamma = \pi_1(M, x_0)$, for $i = 1, \ldots, s$, and we will assume these isomorphisms to be fixed.

Moreover, since M and all $\partial_i M$ are aspherical (this follows from [4]), there is an isomorphism $H_*\left(DCone\left(\cup_{i=1}^s B\Gamma_i \to B\Gamma\right)\right) \to H_*\left(DCone\left(\cup_{i=1}^s \partial_i M \to M\right)\right)$, see [6, Lemma 8].

2.2 Neumann-Yang invariant

The following definitions are given in [7, Section 8] for hyperbolic manifolds (and actually in more generality for representations with certain properties), but they generalize in an obvious way to manifolds of nonpositive sectional curvature, in particular to locally symmetric spaces of noncompact type. (See [2, Definition 8.1 and Definition 8.6] for the definition of the boundary at infinity $\partial_\infty \widetilde{M}$ and of the topology on $\widetilde{M} \cup \partial_\infty \widetilde{M}$.)

Definition 1. *a) Let $\widetilde{M}$ be a simply connected Riemannian manifold of nonpositive sectional curvature and $\partial_\infty \widetilde{M}$ its ideal boundary. Let $C_q\left(\partial_\infty \widetilde{M}\right)$ be the free abelian group freely generated by the (q+1)-tuples of points of $\partial_\infty \widetilde{M}$ modulo the relations*
i) $(z_0, \ldots, z_q) = sign(\tau)\left(z_{\tau(0)}, \ldots, z_{\tau(q)}\right)$ for any permutation τ of $\{0, \ldots, q\}$ and
ii) $(z_0, \ldots, z_q) = 0$ whenever $z_i = z_j$ for some $i \neq j$.

The operator $\partial : C_q\left(\partial_\infty \widetilde{M}\right) \to C_{q-1}\left(\partial_\infty \widetilde{M}\right)$ is defined by $\partial(z_0, \ldots, z_q) = \sum_{i=0}^{q} (-1)^i (z_0, \ldots, \hat{z}_i, \ldots, z_q)$ and linear extension. Let $G = Isom\left(\widetilde{M}\right)$. We define the generalized pre-Bloch group of $\widetilde{M}$ as
$$\mathcal{P}_n\left(\widetilde{M}\right) := H_n\left(C_*\left(\partial_\infty \widetilde{M}\right) \otimes_{\mathbb{Z}G} \mathbb{Z}, \partial \otimes_{\mathbb{Z}G} id\right).$$

b) We define the generalized pre-Bloch groups of $\mathbb{C}$ as
$$\mathcal{P}_n^N(\mathbf{C}) := \mathcal{P}_n\left(SL(N, \mathbf{C})/SU(N)\right).$$

Relation to classical Bloch group. In particular $\mathcal{P}_3^2(\mathbf{C}) = \mathcal{P}_3\left(\mathbf{H}^3\right)$ is the classical pre-Bloch group $\mathcal{P}(\mathbf{C})$ ([7][8]).

One should note that Neumann and Yang [7] considered $C_*\left(\partial_\infty \mathbf{H}^3\right)$ as the free abelian group generated by tuples of *distinct* points in $\partial_\infty \mathbf{H}^3$. We will denote this group by $C_*^{nd}\left(\partial_\infty \mathbf{H}^3\right)$, the superscript standing for 'nondegenerate'. It is easy to see that the inclusion $C_*^{nd}\left(\partial_\infty \mathbf{H}^3\right) \otimes_{\mathbb{Z}G} \mathbb{Z} \to C_*\left(\partial_\infty \mathbf{H}^3\right) \otimes_{\mathbb{Z}G} \mathbb{Z}$ induces an isomorphism in homology. (Here we consider the action of $G := PGL(2, \mathbf{C})$ on $P^1\mathbf{C} = \partial_\infty \mathbf{H}^3$.)

The definition of the pre-Bloch group ([7, Definition 2.1]) is easily seen to be equivalent to $\mathcal{P}(\mathbf{C}) = C_3^{nd}\left(\partial_\infty \mathbf{H}^3\right) \otimes_{\mathbb{Z}G} \mathbb{Z}/\partial C_4^{nd}\left(\partial_\infty \mathbf{H}^3\right) \otimes_{\mathbb{Z}G} \mathbb{Z}$. This agrees with $H_3\left(C_*^{nd}\left(\partial_\infty \mathbf{H}^3\right) \otimes_{\mathbb{Z}G} \mathbb{Z}, \partial \otimes_{\mathbb{Z}G} id\right)$ because $\partial \otimes 1 : C_3^{nd}\left(\partial_\infty \mathbf{H}^3\right) \otimes_{\mathbb{Z}G} \mathbb{Z} \to C_2^{nd}\left(\partial_\infty \mathbf{H}^3\right) \otimes_{\mathbb{Z}G} \mathbb{Z}$ is trivial, see [7, Remark 4.4].

Definition 2. *Let M be a manifold of nonpositive sectional curvature, $\widetilde{M}$ its universal covering with deck group Γ. Let $\sigma \in \overline{S}_q^{str}(M)$ be a proper ideal q-simplex (Definition 4), $\tilde{\sigma} : \Delta^q \to \widetilde{M} \cup \partial_\infty \widetilde{M}$ a lift of σ with vertices $\tilde{\sigma}(v_0), \ldots, \tilde{\sigma}(v_q) \in \partial_\infty \widetilde{M}$. We define the cross-ratio of σ by*
$$cr(\sigma) := (\tilde{\sigma}(v_0), \ldots, \tilde{\sigma}(v_q)) \otimes 1 \in C_q\left(\partial_\infty \widetilde{M}\right) \otimes_{\mathbf{Z}\Gamma} \mathbf{Z}.$$

This is well-defined because all lifts of σ are in the Γ-orbit of $\tilde{\sigma}$.

If G/K is a rank one symmetric space of noncompact type and $\rho : G \to SL(N, \mathbf{C})$ a representation, then upon conjugation we can assume that ρ maps K to $SU(N)$, inducing a smooth map $\rho : G/K \to SL(N, \mathbf{C})/SU(N)$ and a continuous ρ-equivariant map $\rho_\infty : \partial_\infty G/K \to \partial_\infty SL(N, \mathbf{C})/SU(N)$.

In [7], the invariant $\beta(M)$ of hyperbolic 3-manifolds M was defined by means of degree one ideal triangulations. To circumvent the question whether M admits such a triangulation we give an a priori weaker definition and we will show in Section 3 that this one agrees with the Neumann-Yang definition.

The notion of 'proper ideal fundamental cycle' is defined in Definition 6.

Definition 3. *i) Let $\overline{M}$ be a compact, orientable, connected d-manifold with boundary such that $M = \overline{M} - \partial\overline{M}$ is a Riemannian manifold of nonpositive sectional curvature. Let G be the isometry group of the universal covering $\widetilde{M}$.*

Let $\sum_{i=1}^{r} a_i \tau_i$ be a proper ideal fundamental cycle of M. The Neumann-Yang invariant of $\overline{M}$ is defined by

$$\beta\left(\overline{M}\right) := \sum_{i=1}^{r} a_i \left[cr\left(\tau_i\right)\right] \in H_d\left(C_*\left(\partial_\infty \widetilde{M}\right) \otimes_{\mathbb{Z}G} \mathbb{Z}, \partial \otimes_{\mathbb{Z}G} id\right) = \mathcal{P}_d\left(\widetilde{M}\right).$$

ii) If $\widetilde{M} = G/K$ is a rank one symmetric space of noncompact type, and if $\rho : G \to SL(N, \mathbf{C})$ is a representation, then we define

$$\beta_\rho\left(\overline{M}\right) := \rho_*\left(\beta\left(\overline{M}\right)\right) \in \mathcal{P}_d^N\left(\mathbf{C}\right),$$

where $\rho_* : H_*\left(C_*\left(\partial_\infty G/K\right) \otimes_{\mathbb{Z}G} \mathbb{Z}\right) \to H_n\left(C_*\left(\partial_\infty SL(N, \mathbf{C})/SU(N)\right) \otimes_{\mathbb{Z}G} \mathbb{Z}\right)$ is induced by the equivariant chain map $\rho_*\left(c_0, \ldots, c_q\right) = \left(\rho_\infty\left(c_0\right), \ldots, \rho_\infty\left(c_q\right)\right)$.

We will show in Section 3.4 that $\beta(M)$ is well defined, that is, independent of the chosen proper ideal fundamental cycle.

Some results of Neumann-Yang. Let $\overline{M}$ be a compact, orientable, connected d-manifold with boundary such that $M := \overline{M} - \partial\overline{M}$ is a Riemannian manifold of negative sectional curvature *and finite volume*. In [7] it was shown that there is an isomorphism $H_d\left(\overline{M}, \partial\overline{M}\right) \simeq H_d\left(C_*\left(\partial_\infty \widetilde{M}\right)_\Gamma\right)$ for $\Gamma = \pi_1 M$ acting by deck transformations. Although this was stated in [7] for hyperbolic 3-manifolds, the proof in [7] only requires that M is a negatively curved manifold of dimension $d \geq 3$, as we shall indicate now. (This isomorphism will be used in the proof of well-definedness of $\beta(M)$.)

The reason the argument from [7] works in this general context is the following elementary fact about discrete isometry groups in negative curvature:

If $\widetilde{M}$ is a simply connected manifold of negative sectional curvature, if $\Gamma \subset Isom\left(\widetilde{M}\right)$ is discrete of finite covolume, and $x \in \partial_\infty \widetilde{M}$ is not a cusp of Γ, then $Stab(x) = \{\gamma \in \Gamma : \gamma x = x\}$ is an infinite cyclic group.
Proof: If $Stab(x)$ contained loxodromic isometries γ_1, γ_2 with $Fix(\gamma_1) = \{x, y_1\}$ and $Fix(\gamma_2) = \{x, y_2\}$ for $y_1 \neq y_2$, then $[\gamma_1, \gamma_2] \in \Gamma$ would be a parabolic isometry with fixed point x, contradicting the assumption that x is not a cusp of Γ. Hence all elements of $Stab(x)$ have a common fixed point $y \neq x$ and fix the geodesic from x to y. Thus $Stab(x)$ is a discrete subgroup of $\mathbf{R}$, hence infinite cyclic.

The isomorphism $\mathbf{Z} = H_d\left(\overline{M}, \partial\overline{M}\right) \to H_d\left(C_*\left(\partial_\infty\widetilde{M}\right)_\Gamma\right)$ will be defined[3] as the composition

$$H_d\left(\overline{M}, \partial\overline{M}\right) \simeq H_d\left(\Gamma, \mathcal{C}\right) \to H_d\left(\Gamma, \partial_\infty\widetilde{M}\right) \to H_d\left(C_*\left(\partial_\infty\widetilde{M}\right)_\Gamma\right).$$

The first isomorphism can be derived by the same argument as in the proof of [3, Theorem V.1.3]. Namely, $C_*\left(\widetilde{M}\right)/C_*\left(\partial\widetilde{M}\right)$ is acyclic except in degree 1, where its cokernel is $K_* := \mathbb{Z}\mathcal{C}$. Thus $H_*\left(\Gamma, \mathcal{C}\right) = H_*\left(C_*\left(\widetilde{M}\right)/C_*\left(\partial\widetilde{M}\right) \otimes_{\mathbb{Z}G} \mathbb{Z}\right)$. On the other hand, by definition of relative homology, $H_*\left(C_*\left(\widetilde{M}\right)/C_*\left(\partial\widetilde{M}\right) \otimes_{\mathbb{Z}G} \mathbb{Z}\right) = H_*\left(\overline{M}, \partial\overline{M}\right)$.

Lemma 1. $H_d\left(\Gamma, \mathcal{C}\right) \to H_d\left(\Gamma, \partial_\infty\widetilde{M}\right)$ *is an isomorphism for* $d \geq 3$.

In particular, if $\mathcal{C} = \emptyset$, then $H_d\left(\Gamma; \mathbf{Z}\right) \to H_d\left(\Gamma, \partial_\infty\widetilde{M}\right)$ *is an isomorphism for* $d \geq 3$.

Proof: ([7, Section 3]) By Shapiro's Lemma, $H_i\left(\Gamma, \mathbb{Z}\left(\partial_\infty\widetilde{M} - \mathcal{C}\right)\right)$ is the direct sum of H_i of the isotropy groups for the orbits of Γ on $\partial_\infty\widetilde{M} - \mathcal{C}$. (This is also true if $\mathcal{C} = \emptyset$.)

The homology of a cyclic group vanishes for $i \geq 2$, hence $H_i\left(\Gamma, \mathbb{Z}\left(\partial_\infty\widetilde{M} - \mathcal{C}\right)\right) = 0$ for $i \geq 2$. The long exact homology sequence of $J\mathcal{C} \to J\partial_\infty\widetilde{M} \to \mathbb{Z}\left(\partial_\infty\widetilde{M} - \mathcal{C}\right)$ implies then the desired isomorphism for $d \geq 3$. $\qquad QED$

Lemma 2. $H_d\left(\Gamma, \partial_\infty\widetilde{M}\right) \to H_d\left(C_*\left(\partial_\infty\widetilde{M}\right)_\Gamma\right)$ *is an isomorphism.*

Proof: ([7, Proposition 3.2]) Consider the spectral sequence with E^1-term $H_i\left(\Gamma, C_{j-1}\left(\partial_\infty\widetilde{M}\right)\right)$ that converges to $H_*\left(\Gamma, \partial_\infty\widetilde{M}\right)$. The isotropy group of a pair of points is infinite cyclic or trivial, that is of homological dimension at most 1, hence $E^1_{p,q} = 0$ if $q \geq 2$. Thus the only nontrivial d_1 is $d_1 : H_1\left(\Gamma, C_1\left(\partial_\infty\widetilde{M}\right)\right) \to H_1\left(\Gamma, C_0\left(\partial_\infty\widetilde{M}\right)\right)$. In [7, Lemma 3.3] it is proven that d_1 is injective for hyperbolic 3-manifolds. The proof uses only the fact that stabilizers of points and stabilizers of pairs of points in $\partial_\infty\widetilde{M}$ are infinitely cyclic or trivial. By the observation above this is true for any negatively curved manifold, hence the proof of [7] applies verbatim. $\qquad QED$

Corollary 1. *If* $\widetilde{M}$ *is a simply connected manifold of negative sectional curvature and dimension* $d \geq 3$, *and if* $\Gamma \subset Isom^+\left(\widetilde{M}\right)$ *is a discrete subgroup of finite covolume, then* $H_d\left(C_*\left(\partial_\infty\widetilde{M}\right)_\Gamma\right) \simeq \mathbf{Z}$.

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[3]$\mathcal{C}$ is the set of cusps of Γ. If $\mathcal{C}$, then $H_*\left(\Gamma, \mathcal{C}\right)$ is understood to be $H_*\left(\Gamma, \mathbf{Z}\right)$, that is in this case we claim an isomorphism $H_*\left(M; \mathbf{Z}\right) \simeq H_*\left(\Gamma; \mathbf{Z}\right) \simeq H_*\left(\Gamma, \partial_\infty\widetilde{M}\right)$.

Recall that, for a G-set Ω, $H_*\left(G, \Omega\right)$ is defined as the homology of $P_* \otimes_{\mathbb{Z}G} \mathbb{Z}$, where

$$\ldots P_3 \to P_2 \to P_1 \to J\Omega$$

is a $\mathbb{Z}G$-projective resolution of $J\Omega := ker\left(\mathbb{Z}\Omega \to \mathbb{Z}\right)$, the kernel of the augmentation map.

3 Ideal fundamental cycles

3.1 Definition

We refer to [2] for basic notions about simply connected manifolds $\widetilde{M}$ of nonpositive sectional curvature, especially for the notion of ideal boundary $\partial_\infty \widetilde{M}$ ([2, Definition 8.1]) and the topology on $\widetilde{M} \cup \partial_\infty \widetilde{M}$ ([2, Definition 8.5]). For each ordered tuple $(v_0, \ldots, v_q) \in \left(\widetilde{M} \cup \partial_\infty \widetilde{M} \right)^{q+1}$, there is a unique straight ideal simplex $str\,(v_0, \ldots, v_q)$ with these vertices. We call this simplex a *genuine straight simplex* resp. a *proper ideal straight simplex* if all vertices belong to $\widetilde{M}$ resp. all vertices belong to $\partial_\infty \widetilde{M}$. In the sequel, Δ^n is the standard n-simplex and Δ_0^n its set of vertices.

Definition 4. *Let M be a Riemannian manifold of nonpositive sectional curvature, $\widetilde{M}$ its universal cover, Γ a discrete group of isometries of $\widetilde{M}$ such that $M = \Gamma \backslash \widetilde{M}$, $\pi : \widetilde{M} \to M$ the canonical projection.*
a) Let $\overline{C}_n^{str}(M)$ be the free abelian group generated by

$$\overline{S}_n^{str}(M) := \left\{ \sigma : \Delta^n - \Delta_0^n \to M : \quad \begin{array}{c} \text{there is an ideal straight simplex } \tilde{\sigma} : \Delta^n \to \widetilde{M} \cup \partial_\infty \widetilde{M} \\ \text{with } \pi \circ \tilde{\sigma} \mid_{\Delta^n - \Delta_0^n} = \sigma \mid_{\Delta^n - \Delta_0^n} \end{array} \right\}.$$

A simplex $\sigma \in \overline{S}_n^{str}(M)$ will be called genuine resp. proper ideal if (some, hence each) $\tilde{\sigma}$ is a genuine resp. a proper ideal straight simplex in $\widetilde{M}$.
b) For $x_0 \in M$ define $C_*^{str,x_0}(M) \subset \overline{C}_*^{str}(M)$ to be the subcomplex generated by genuine straight simplices with all vertices in x_0.
c) For $c_0 \in \partial_\infty \widetilde{M}$ define $\overline{C}_*^{str,c_0}\left(\widetilde{M} \right) \subset \overline{C}_*^{str}(M)$ to be the subcomplex generated by proper ideal straight simplices such that (some, hence each) $\tilde{\sigma}$ has all vertices in Γc_0.

There is a chain isomorphism $EM : C_*^{simp}(B\Gamma) \to C_*^{str,x_0}(M)$, essentially due to Eilenberg-MacLane (cf. Section 2.1 in [6]).

Definition 5. *Let M be an orientable smooth d-manifold. For $\sigma \in \overline{S}_d^{str}(M)$ and $y \in M$ we define $deg_y(\sigma) = 0$ if $y \notin \sigma\left(\Delta^d - \partial \Delta^d \right)$, else we define*

$$deg_y(\sigma) := \sum_{\sigma(x)=y} sign\,(det\,(D_x\sigma)).$$

For a chain $z = \sum_{i=1}^r a_i \sigma_i \in \overline{C}_d^{str}(M)$ we define

$$deg_y(z) := \sum_{i=1}^r a_i deg_y(\sigma_i).$$

Homology invariance of *deg* for closed manifolds. If $z = \sum_{i=1}^r a_i \sigma_i \in \overline{C}_d^{str}(M)$, then it is a consequence of the Sard Lemma that almost every $y \in M$ is not contained in $\cup_{i=1}^r \sigma_i\left(\partial \Delta^d \right)$. If M is a closed, orientable, connected d-manifold, $\partial z = 0$ and all σ_i are genuine straight simplices, then for all $y \notin \cup_{i=1}^r \sigma_i\left(\partial \Delta^d \right)$ the isomorphism

$H_d\left(M, M - \{y\}\right) \to \mathbf{Z}$ sends the relative homology class of z to $deg_y\left(z\right)$. In particular $deg_y\left(z\right)$ depends only on the homology class $[z] \in H_d\left(M\right)$.

Since $deg_y\left(z\right)$ is the same for almost all $y \in M$ we will occasionally just write $deg\left(z\right)$.

Observation 1. *a) Let M be a closed, orientable, connected Riemannian d-manifold of nonpositive sectional curvature. Let $z = \sum_{i=1}^{r} a_i \sigma_i \in \overline{C}_d^{str}\left(M\right)$ be a singular cycle consisting of genuine straight simplices. If z represents the fundamental class $[M]$, then $deg_y\left(z\right) = 1$ for all $y \notin \cup_{i=1}^{r} \sigma_i \left(\partial \Delta^d\right)$.*
b) Let $z = \sum_{i=1}^{r} \sigma_i \in \overline{C}_d^{str}\left(M\right)$ be an ideal degree one triangulation of a finite volume hyperbolic 3-manifold as in [7]. Then $deg_y\left(z\right) = 1$ for all $y \notin \cup_{i=1}^{r} \sigma_i \left(\partial \Delta^d\right)$.

Proof: Smooth closed manifolds are triangulable by Whitehead's Theorem, by orientability the simplices of the triangulation can be coherently oriented. The resulting cycle z represents $[M]$ and clearly satisfies $deg\left(z\right) = 1$. Since homologous cycles have the same degree, a) follows. b) is part of the definition in [7, Section 2.1]. *QED*

This motivates the following definition.

Definition 6. *Let M be an orientable Riemannian d-manifold of nonpositive sectional curvature, $\widetilde{M}$ its universal cover. We say that $z = \sum_{i=1}^{r} a_i \sigma_i \in \overline{C}_d^{str}\left(M\right)$ is an ideal fundamental cycle if*
a) $\partial z = 0$, and
b) $deg_y\left(z\right) = 1$ for all $y \notin \cup_{i=1}^{r} \sigma_i \left(\partial \Delta^d\right)$.
An ideal fundamental cycle is said to be proper ideal if all σ_i are proper.

3.2   Mapping fundamental cycles to ideal fundamental cycles

Lemma 3. *Let $\widetilde{M}$ be a simply connected Riemannian manifold of nonpositive sectional curvature, Γ a discrete group of isometries of $\widetilde{M}$, $M = \Gamma \backslash \widetilde{M}, \pi : \widetilde{M} \to M$ the projection, $\tilde{x}_0 \in \widetilde{M}, x_0 = \pi\left(\tilde{x}_0\right), c_0 \in \partial_\infty \widetilde{M}$.*

Then there exists a Γ-invariant chain map $\widetilde{\Psi} : \overline{C}_*^{str}\left(\widetilde{M}\right) \to \overline{C}_*^{str}\left(\widetilde{M}\right)$ which is given on 0-simplices by

$$\widetilde{\Psi}\left(x\right) = x \text{ if } x \in \left(\widetilde{M} - \Gamma \tilde{x}_0\right) \cup \partial_\infty \widetilde{M},$$

$$\widetilde{\Psi}\left(\gamma \tilde{x}_0\right) = \gamma c_0 \text{ for all } \gamma \in \Gamma,$$

such that $\widetilde{\Psi}$ descends to a chain map $\Psi : \overline{C}_*^{str,x_0}\left(M\right) \to \overline{C}_*^{str,c_0}\left(M\right)$.

If M is a closed, orientable, connected manifold then $\Psi\left(z\right)$ is an ideal fundamental cycle whenever $z \in \overline{C}_*^{str,x_0}\left(M\right)$ is an ideal fundamental cycle.

Proof: We have defined $\widetilde{\Psi}$ Γ-equivariantly on the 0-skeleton. Since straight simplices in $\widetilde{M} \cup \partial_\infty \widetilde{M}$ are determined by their vertices, this extends uniquely to a Γ-equivariant chain map $\widetilde{\Psi}$. Since $\widetilde{\Psi}$ is Γ-equivariant, it descends to a chain map on $\overline{C}_*^{str}\left(M\right)$. By construction, its restriction to $\overline{C}_*^{str,x_0}\left(M\right)$ has image in $\overline{C}_*^{str,c_0}\left(M\right)$.

To prove the last claim, since deg depends only on the homology class, it suffices to prove $deg\left(\Psi\left(z\right)\right) = 1$ for some ideal fundamental cycle z. Choose some fundamental cycle with all vertices in some point $x \neq x_0$, then $\Psi\left(z\right) = z$, which implies $deg\left(\Psi\left(z\right)\right) = 1$.

QED

We remark that $\Psi : \overline{C}_*^{str,x_0}(M) \to \overline{C}_*^{str,c_0}(M)$ is a chain isomorphism if and only if Γ acts freely on Γc_0, i.e., if and only if $c_0 \in \partial_\infty \widetilde{M}$ is not a fixed point of any $\gamma \in \Gamma$.

3.3   Cusped manifolds

Assumption A: *Let $\overline{M}$ be a manifold with boundary components $\partial_1 \overline{M}, \ldots, \partial_s \overline{M}$, $M = \overline{M} - \partial \overline{M}$.*

Assume that M is a Riemannian manifold of nonpositive sectional curvature of finite volume and that the universal covering $\widetilde{M}$ is a visibility manifold ([2, Definition 9.28]). Let $\pi : \widetilde{M} \to M$ be the canonical projection, and let $\widetilde{x_0} \in \widetilde{M}, x_0 = \pi(\widetilde{x}_0)$, and $c_0 \in \partial_\infty \widetilde{M}$ be fixed.

Let $x_i \in \partial_i \overline{M}$ for $i = 1, \ldots, s$ and fix (using a path from x_0 to x_i) an identification of $\Gamma_i := \pi_1 \left(\partial_i \overline{M}, x_i \right)$ with a subgroup of $\Gamma := \pi_1 \left(\overline{M}, x_0 \right) = \pi_1(M, x_0)$ (which acts on $\widetilde{M}$).

Remark: A rank one symmetric space of noncompact type is a visibility manifold. If Assumption A holds, then it follows from [4] that there are $c_1, \ldots, c_s \in \partial_\infty \widetilde{M}$ such that we have a continuous projection

$$p : \widetilde{M} \bigcup \cup_{i=1}^s \Gamma c_i \to DCone \left(\cup_{i=1}^s \partial_i \overline{M} \to \overline{M} \right)$$

with $x \in \Gamma c_i \iff p(x)$ is the cone point of $Cone \left(\partial_i \overline{M} \right)$, for $i = 1, \ldots, s$ (see [6, Section 4.4]).

Definition 7. *If Assumption A holds, then a simplex $\sigma \in C_* \left(DCone \left(\cup_{i=1}^s \partial_i \overline{M} \to \overline{M} \right) \right)$ is said to be straight if some (hence any) lift $\tilde{\sigma} \in C_* \left(\widetilde{M} \bigcup \cup_{i=1}^s \Gamma c_i \right) \subset C_* \left(\widetilde{M} \cup \partial_\infty \widetilde{M} \right)$ with $p(\tilde{\sigma}) = \sigma$ is a straight simplex.*

If a vertex of $\tilde{\sigma}$ is in γc_i for some $\gamma \in \Gamma, 1 \le i \le s$, then we call the corresponding vertex of $\sigma = p(\tilde{\sigma})$ an ideal vertex. All other vertices of σ are called interior vertices. Let

$$\widehat{C}_*^{str,x_0}(M) \subset C_* \left(DCone \left(\cup_{i=1}^s \partial_i \overline{M} \to \overline{M} \right) \right)$$

be the subcomplex freely generated by the straight simplices for which
- either all vertices are in x_0,
- or the last vertex is in Γc_i for some $1 \le i \le s$, all other vertices are in x_0, and the homotopy classes of all edges between interior vertices belong to the image of Γ_i in Γ.

$$\widehat{C}_*^{str,c_0}(M) \subset C_* \left(DCone \left(\cup_{i=1}^s \partial_i \overline{M} \to \overline{M} \right) \right)$$

is the subcomplex freely generated by simplices $\tau = \pi(\tilde{\tau})$ such that
- either all vertices of $\tilde{\tau}$ are in Γc_0,
- or the last vertex of $\tilde{\tau}$ is in Γc_i for some $1 \le i \le s$ and all other vertices of $\tilde{\tau}$ are in Γc_0.

Lemma 4. *([6], Lemma 8a) If Assumption A holds, then*

$$\widehat{C}_*^{str,x_0}(M) \cong C_*^{simp} \left(DCone \left(\cup_{i=1}^s B\Gamma_i \to B\Gamma \right) \right).$$

9

Homology invariance of *deg* for cusped manifolds. If $\overline{M}$ is a compact, orientable, connected d-manifold with boundary and $z = \sum_{i=1}^r a_i \sigma_i \in \overline{C}_d^{str}\left(DCone\left(\cup_{i=1}^s \partial_i \overline{M} \to \overline{M}\right)\right)$ satisfies $\partial z = 0$, then for all $y \notin \cup_{i=1}^r \sigma_i\left(\partial \Delta^d\right)$ the isomorphism

$$H_d\left(DCone\left(\cup_{i=1}^s \partial_i \overline{M} \to \overline{M}\right), DCone\left(\cup_{i=1}^s \partial_i \overline{M} \to \overline{M}\right) - \{y\}\right) \to \mathbf{Z}$$

sends the relative homology class of z to $deg_y(z)$. In particular $deg_y(z)$ depends only on the homology class $[z] \in H_d\left(DCone\left(\cup_{i=1}^s \partial_i \overline{M} \to \overline{M}\right)\right) = H_d\left(\overline{M}, \partial \overline{M}\right)$.

Lemma 5. *If Assumption A holds, then there is a chain map*

$$\hat{\Psi} : \hat{C}_*^{str,x_0}(M) \to \hat{C}_*^{str,c_0}(M),$$

such that the restriction of $\hat{\Psi}$ to C_*^{str,x_0} is the chain map Ψ defined by Lemma 3.

If $z \in \hat{C}_*^{str,x_0}(M)$ is an ideal fundamental cycle, then $\hat{\Psi}(z)$ is an ideal fundamental cycle.

Proof: If $\sigma \in \hat{C}_*^{str,x_0}(M)$ has all vertices in x_0, that is if $\sigma \in C_*^{str,x_0}(M)$, then we let $\hat{\Psi}(\sigma) := \Psi(\sigma)$, where Ψ is defined by Lemma 3.

In the other case σ lifts to a q-simplex $\tilde{\sigma} : \Delta^q \to \widetilde{M} \cup \partial_\infty \widetilde{M}$ whose last vertex v_q is γc_i for some $\gamma \in \Gamma, 1 \leq i \leq s$ and we have $\partial_q \sigma \in C_{n-1}^{str,x_0}(M)$. Then we define $\hat{\Psi}(\sigma) := p(\tilde{\tau}) \in C_n^{str,c_0}(M)$, where $\tilde{\tau}$ is the unique straight q-simplex with last vertex γc_0 and with $\partial_q \tilde{\tau} = \widetilde{\Psi}(\partial_q \tilde{\sigma})$.

Ψ is a chain map, hence $\hat{\Psi}(\partial \sigma) = \partial \hat{\Psi}(\sigma)$ whenever all vertices of σ are interior vertices. In the other case, if σ lifts to $\tilde{\sigma}$ with $v_q = \gamma c_i$, then we have $\hat{\Psi}(\partial_q \sigma) = \Psi(\partial_q \sigma) = \partial_q \Psi(\sigma) = \partial_q \hat{\Psi}(\sigma)$ and, for $0 \leq j \leq q-1$, $\hat{\Psi}(\partial_j \sigma) = p(\tilde{\kappa})$, where $\tilde{\kappa}$ is the straight $q-1$-simplex with last vertex γc_0 and with $\partial_{q-1} \tilde{\kappa} = \widetilde{\Psi}(\partial_j \partial_q \tilde{\sigma}) = \partial_j \widetilde{\Psi}(\partial_q \tilde{\sigma})$, hence $p(\tilde{\kappa}) = \partial_j \hat{\Psi}(\sigma)$. Thus $\hat{\Psi}$ is a chain map.

Again, since *deg* depends only on the homology class, it suffices to prove $deg\left(\hat{\Psi}(z)\right) = 1$ for some ideal fundamental cycle z. Choose some ideal fundamental cycle with all *interior* vertices in some point $x \neq x_0$, then $\hat{\Psi}(z) = z$, which implies $deg\left(\hat{\Psi}(z)\right) = 1$.

$$QED$$

As a side remark, unrelated to the rest of the paper, we observe that a construction analogous to the proof of Lemma 5 proves the possibility of an alternative definition of the simplicial volume for hyperbolic manifolds. The simplicial volume of a compact, orientable, connected manifold $\overline{M}$, as defined by Gromov, is $\parallel \overline{M}, \partial \overline{M} \parallel = inf\left\{\sum_{i=1}^r \mid a_i \mid : \sum_{i=1}^r a_i \sigma_i \text{ represents } \left[\overline{M}, \partial \overline{M}\right]\right\}$. If we define the ideal simplicial volume as $\parallel M \parallel_{ideal} = inf\left\{\sum_{i=1}^r \mid a_i \mid : \sum_{i=1}^r a_i \tau_i \text{ is an ideal fundamental cycle}\right\}$, then we can apply a similar construction to map any relative fundamental cycle to an ideal fundamental cycle, thus proving $\parallel M \parallel_{ideal} \leq \parallel \overline{M}, \partial \overline{M} \parallel$ whenever $M = \overline{M} - \partial \overline{M}$ is non-positively curved. Moreover, if M is hyperbolic, then each ideal simplex has volume at most V_n, thus the ideal simplicial volume can not be bigger than $\frac{1}{V_n} Vol(M)$. Since $\parallel \overline{M}, \partial \overline{M} \parallel = \frac{1}{V_n} Vol(M)$ by the Gromov-Thurston Theorem, this implies

$$\parallel \overline{M}, \partial \overline{M} \parallel = \parallel M \parallel_{ideal}$$

for hyperbolic manifolds of finite volume.

3.4 Well-definedness of $\beta(M)$

If $\overline{M}$ is a compact, orientable, connected manifold such that its interior admits a metric of negative sectional curvature, then there exists an ideal fundamental cycle. Indeed, one can take some triangulation $\sum_{i=1}^r \tau_i$ of $(\overline{M}, \partial \overline{M})$, let $\sum_{j=1}^p \kappa_j := DCone(\partial(\sum_{i=1}^r \tau_i))$ be the cone over the induced triangulations of the path-components $\partial_1 M, \ldots, \partial_s M$ of ∂M, with one cone point for each path-component $\partial_i M$, and then use the homeomomorphism $DCone(\cup_{l=1}^s \partial_i M \to M) \cong \Gamma \backslash G/K \cup \{\Gamma c_1, \ldots, \Gamma c_s\}$.

Thus we can define $\beta(\overline{M})$ by Definition 3. We will now prove that the definition of $\beta(\overline{M})$ does not depend on the chosen proper ideal fundamental cycle.

Lemma 6. *Let $\overline{M}$ be a compact, orientable, connected manifold such that its interior admits a metric of negative sectional curvature and finite volume. Let G be the isometry group of $\widetilde{M}$. Then $\beta(\overline{M})$ in Definition 3 is well-defined: if $\sum_{i=1}^r a_i \tau_i$ and $\sum_{j=1}^s b_j \kappa_j$ are proper ideal fundamental cycles, then $\sum_{i=1}^r a_i [cr(\tau_i)] = \sum_{j=1}^s b_j [cr(\kappa_j)] \in H_*\left(C_*\left(\partial_\infty \widetilde{M}\right)_G\right)$.*

Proof: By assumption and Definition 6, we have $\sum_{i=1}^r a_i deg_x(\tau_i) = \sum_{j=1}^s b_j deg_x(\kappa_j) = 1$ for all $x \notin \cup_{i=1}^r \tau_i(\partial \Delta^n) \cup \cup_{j=1}^s \kappa_j(\partial \Delta^n)$.

If we define algebraic volume $algvol(\sigma)$ of straight simplices as in [1], p.107, by $algvol(\sigma) = \pm vol(\sigma)$ with the sign according to the orientation of σ, then[4] (as in [1], p.109):

$$\sum_{i=1}^r a_i algvol(\tau_i) = \sum_{i=1}^r a_i \int_M \sum_{\tau_i(x)=y} sign(D_x \tau_i) dvol(y)$$

$$= \int_M \sum_{i=1}^r a_i deg_y(\tau_i) dvol(y) = \int_M 1 dvol(y) = vol(M),$$

in particular, $\sum_{i=1}^r a_i algvol(\tau_i) = \sum_{j=1}^s b_j algvol(\kappa_j)$.

$$Algvol : C_*\left(\partial_\infty \widetilde{M}\right)_G \to \mathbf{R}$$

$$(c_0, \ldots, c_n) \to algvol(str(c_0, \ldots, c_n)),$$

where $str(c_0, \ldots, c_n)$ is the unique proper ideal straight simplex with these vertices, is by Stokes Theorem a (nontrivial) chain map and satisfies of course $Algvol \circ cr = algvol$.

By Corollary 1 we have $H_*\left(C_*\left(\partial_\infty \widetilde{M}\right)_\Gamma\right) \simeq \mathbf{Z}$, therefore $Algvol$ must be injective on $H_*\left(C_*\left(\partial_\infty \widetilde{M}\right)_\Gamma\right)$.

In particular,

$$\sum_{i=1}^r a_i algvol(cr(\tau_i)) = \sum_{j=1}^s b_j algvol(cr(\kappa_j))$$

implies

$$\sum_{i=1}^r a_i [cr(\tau_i)] = \sum_{j=1}^s b_j [cr(\kappa_j)] \in H_*\left(C_*\left(\partial_\infty \widetilde{M}\right)_\Gamma\right) \simeq \mathbf{Z},$$

---

[4]using that $\cup_{i=1}^r \tau_i(\partial \Delta^n) \cup \cup_{j=1}^s \kappa_j(\partial \Delta^n)$ is a null set, thus can be neglected for integration

hence

$$\sum_{i=1}^{r} a_i \left[cr\left(\tau_i\right) \right] = \sum_{j=1}^{s} b_j \left[cr\left(\kappa_j\right) \right] \in H_* \left(C_* \left(\partial_\infty \widetilde{M} \right)_G \right).$$

<div align="right">QED</div>

4 Evaluation

Let $\widetilde{M} = G/K$ be a symmetric space of noncompact type. Fix some $c_0 \in \partial_\infty \widetilde{M}$. We define the evaluation map

$$ev_{G,c_0} : C_*^{simp}\left(BG\right) \to C_* \left(\partial_\infty \widetilde{M} \right) \otimes_{\mathbb{Z}G} \mathbb{Z}$$

on generators by

$$ev_G\left(g_1, \ldots, g_n\right) = \left(c_0, g_1 c_0, g_1 g_2 c_0, \ldots, g_1 g_2 \ldots g_n c_0\right) \otimes 1.$$

It is straightforward to check that ev_{G,c_0} extends linearly to a chain map, thus it induces a homomorphism

$$ev_{G,c_0 *} : H_*\left(G\right) \to \mathcal{P}_*\left(G/K\right).$$

If $\Gamma_1, \ldots, \Gamma_s$ is a set of subgroups of G such that for $1 \le i \le s$ there is some $c_i \in \partial_\infty \widetilde{M}$ with $\Gamma_i \subset Stab\left(c_i\right)$, then we define

$$ev_{G,\Gamma_1,\ldots,\Gamma_s,c_0,c_1,\ldots,c_s} : C_*^{simp}\left(DCone\left(\cup_{i=1}^{s} B\Gamma_i \to BG\right)\right) \to C_* \left(\partial_\infty \widetilde{M} \right) \otimes_{\mathbb{Z}G} \mathbb{Z}$$

by

$$ev_{G,\Gamma_1,\ldots,\Gamma_s,c_0,c_1,\ldots,c_s}\left(g_1, \ldots, g_n\right) = ev_G\left(g_1, \ldots, g_n\right)$$

if $\left(g_1, \ldots, g_n\right) \in C_*^{simp}\left(BG\right)$ and by

$$ev_{G,\Gamma_1,\ldots,\Gamma_s,c_0,c_1,\ldots,c_s}\left(Cone\left(\gamma_1, \ldots, \gamma_{n-1}\right)\right) = \left(c_0, \gamma_1 c_0, \gamma_1 \gamma_2 c_0, \ldots, \gamma_1 \gamma_2 \ldots \gamma_{n-1} c_0, c_i\right) \otimes 1$$

if $Cone\left(\gamma_1, \ldots, \gamma_{n-1}\right) \in C_*^{simp}\left(Cone\left(B\Gamma_i\right)\right)$ for some $1 \le i \le s$.

The assumption $\Gamma_i \subset Stab\left(c_i\right)$ implies that $ev_{G,\Gamma_1,\ldots,\Gamma_s,c_0,c_1,\ldots,c_s}$ is a chain map, as can be shown by a straightforward calculation. Indeed, for $j \ne 0$ it is immediate from the definition that $\partial_j ev\left(Cone\left(\gamma_1, \ldots, \gamma_{n-1}\right)\right) = ev\left(\partial_j\left(Cone\left(\gamma_1, \ldots, \gamma_{n-1}\right)\right)\right)$, and for $j = 0$ we have (omitting the indices of ev)

$$\partial_0 ev\left(Cone\left(\gamma_1, \ldots, \gamma_{n-1}\right)\right) = \left(\gamma_1 c_0, \gamma_1 \gamma_2 c_0, \ldots, \gamma_1 \ldots \gamma_{n-1} c_0, c_i\right) \otimes 1$$

$$= \left(c_0, \gamma_2 c_0, \ldots, \gamma_2 \ldots \gamma_{n-1} c_0, \gamma_1^{-1} c_i\right) \otimes 1 = \left(c_0, \gamma_2 c_0, \ldots, \gamma_2 \ldots \gamma_{n-1} c_0, c_i\right) \otimes 1$$

$$= ev\left(Cone\left(\gamma_2, \ldots, \gamma_{n-1}\right)\right) = ev\left(\partial_0 Cone\left(\gamma_1, \ldots, \gamma_{n-1}\right)\right),$$

where the second equality uses the definition of the tensor product (and that G acts trivially on $\mathbf{Z}$), and the third equality is true because of $\gamma_1 \in \Gamma_i \subset Stab\left(c_i\right)$.

Theorem 1. *Let $\overline{M}$ be a compact, orientable, connected manifold with boundary such that $M := \overline{M} - \partial\overline{M} = \Gamma\backslash G/K$ is a finite-volume, locally rank one symmetric space of noncompact type. Let $\rho: G \to GL(N,\mathbb{C})$ be a representation. For $1 \le i \le s$, let Γ_i be the subgroup of $\Gamma \simeq \pi_1(M,x_0)$ corresponding to $\pi_1(\partial_i M, x_i)$, let $\Gamma'_i := \rho(\Gamma_i)$ and fix some $c_0 \in \partial_\infty\widetilde{M}$ and $c_i \in Stab(\Gamma_i) \subset \partial_\infty\widetilde{M}$ for $1 \le i \le s$.*
   Then
$$ev_{SL(N,\mathbf{C}),\Gamma'_1,\ldots,\Gamma'_s,\rho_\infty(c_0),\rho_\infty(c_1),\ldots,\rho_\infty(c_s)*}\left(\overline{\gamma}\left(\overline{M}\right)\right) = \beta_\rho\left(\overline{M}\right).$$

Proof: Since M is a finite-volume locally rank one symmetric space, Γ_i must be unipotent. Then Γ'_i is unipotent and $\gamma(M)$ is defined by Proposition 1. Moreover $c_i \in Stab(\Gamma_i) \subset \partial_\infty\widetilde{M}$ exists by the Remark after Assumption A in Section 3.3.

Since G is semisimple, ρ is actually a representation $\rho: G \to SL(N,\mathbf{C})$.

By definition of $ev_{SL(N,\mathbf{C})\Gamma'_1,\ldots,\Gamma'_s}$ the upper square of the diagram commutes, and ρ-equivariance of ρ_∞ implies easily that the second square (whose vertical arrows are induced by $B\rho$ resp. ρ_∞) commutes.

(For the sake of readability of the diagram we omit the indices $G, \Gamma_1, \ldots, \Gamma_s, c_0, c_1, \ldots, c_s$ resp. $SL(N,\mathbf{C}), \Gamma'_1, \ldots, \Gamma'_s, \rho_\infty(c_0), \rho_\infty(c_1), \ldots, \rho_\infty(c_s)$ from the notation.)

$$
\begin{array}{ccc}
C_*\left(BSL\left(N,\mathbf{C}\right)\right) & \xrightarrow{\ ev\ } & C_*\left(\partial_\infty\left(SL\left(N,\mathbf{C}\right)/SU\left(N\right)\right)\right)_{SL(N,\mathbf{C})} = \mathcal{P}^N_*\left(\mathbf{C}\right) \\
\downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle =} \\
C_*\left(DCone\left(\cup_{i=1}^s B\Gamma'_i \to BSL\left(N,\mathbf{C}\right)\right)\right) & \xrightarrow{\ ev\ } & C_*\left(\partial_\infty\left(SL\left(N,\mathbf{C}\right)/SU\left(N\right)\right)\right)_{SL(N,\mathbf{C})} = \mathcal{P}^N_*\left(\mathbf{C}\right) \\
\uparrow{\scriptstyle B\rho} & & \uparrow{\scriptstyle \rho_*} \\
C_*\left(DCone\left(\cup_{i=1}^s B\Gamma_i \to BG\right)\right) & \xrightarrow{\ ev\ } & C_*\left(\partial_\infty\widetilde{M}\right)_G = \mathcal{P}_*\left(\widetilde{M}\right) \\
\uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle i_2} \\
C_*\left(DCone\left(\cup_{i=1}^s B\Gamma_i \to B\Gamma\right)\right) & \xrightarrow{\ ev\ } & C_*\left(\partial_\infty\widetilde{M}\right)_\Gamma \\
\uparrow{\scriptstyle \hat\Phi} & & \uparrow{\scriptstyle cr} \\
\hat{C}^{str,x_0}_*\left(M\right) & \xrightarrow{\ \hat\Psi\ } & \hat{C}^{str,c_0}_*\left(M\right) \\
\uparrow{\scriptstyle str} & & \\
C_*\left(M \cup \{\Gamma c_1, \ldots, \Gamma c_s\}\right) & & \\
\uparrow{\scriptstyle \simeq} & & \\
Z_*\left(\overline{M}, \partial\overline{M}\right) \to C_*\left(DCone\left(\cup_{i=1}^s \partial_i\overline{M} \to \overline{M}\right)\right) & &
\end{array}
$$

i_1 is the obvious inclusion and $i_2 = id \otimes id$. Thus the third square commutes.

$\hat\Phi$ is the isomorphism given by Lemma 4. If $\sigma \in \hat{C}^{str,x_0}_*(M)$ has all vertices on x_0, then it lifts to a simplex $\tilde\sigma$ in $\widetilde{M}$ with vertices $\tilde{x}_0, \gamma_1\tilde{x}_0, \gamma_1\gamma_2\tilde{x}_0, \ldots, \gamma_1\ldots\gamma_n\tilde{x}_0$ and (by

the proof in [6]) $\hat{\Phi}(\tilde{\sigma}) = (\gamma_1, \gamma_2, \ldots, \gamma_n)$. By Lemma 3, $\widetilde{\Psi}(\tilde{\sigma})$ is a simplex with vertices $c_0, \gamma_1 c_0, \ldots, \gamma_1 \ldots \gamma_n c_0$, thus

$$cr\left(\hat{\Psi}(\sigma)\right) = (c_0, \gamma_1 c_0, \ldots, \gamma_1 \ldots \gamma_n c_0) \otimes_{\mathbf{Z}\Gamma} 1 = ev_{G,\Gamma_1,\ldots,\Gamma_s,c_0,c_1,\ldots,c_s}(\gamma_1, \ldots, \gamma_n) = ev_{G,\Gamma_1,\ldots,\Gamma_s,c_0,c_1,\ldots,c_s}\left(\hat{\Phi}(\sigma)\right).$$

If $\sigma \in \hat{C}_*^{str,x_0}(M)$ has its n-th vertex v_n in Γc_i, then it lifts to a simplex $\tilde{\sigma}$ in $\widetilde{M} \cup \partial_\infty \widetilde{M}$ with vertices $\tilde{x}_0, \gamma_1 \tilde{x}_0, \gamma_1 \gamma_2 \tilde{x}_0, \ldots, \gamma_1 \ldots \gamma_{n-1} \tilde{x}_0, v_n$ and $\hat{\Phi}(\sigma)$ is the cone over $(\gamma_1, \ldots, \gamma_{n-1})$. By Lemma 3, $\widetilde{\Psi}(\tilde{\sigma})$ is a simplex with vertices $c_0, \gamma_1 c_0, \ldots, \gamma_1 \ldots \gamma_{n-1} c_0, c_i$, thus

$$cr\left(\hat{\Psi}(\sigma)\right) = (c_0, \gamma_1 c_0, \ldots, \gamma_1 \ldots \gamma_{n-1} c_0, c_i) \otimes_{\mathbf{Z}\Gamma} 1 = ev_{G,\Gamma_1,\ldots,\Gamma_s,c_0}(Cone(\gamma_1, \ldots, \gamma_{n-1})) = ev_{G,\Gamma_1,\ldots,\Gamma_s,c_0}\left(\hat{\Phi}(\sigma)\right).$$

This proves that the 4th square commutes.

$Z_*\left(\overline{M}, \partial\overline{M}\right) \subset C_*(M, \partial M)$ denotes the group of relative cycles. If $z = \sum_{k=1}^r a_k \tau_k \in Z_*\left(\overline{M}, \partial\overline{M}\right)$, that is if $\partial z = \sum_{j=1}^q b_j \kappa_j \in C_*\left(\partial\overline{M}\right)$, then each κ_j has image in some component $\partial_i \overline{M}$ of $\partial\overline{M}$ and $cone(z) \in C_*\left(DCone\left(\cup_{i=1}^s \partial_i\overline{M} \to \overline{M}\right)\right)$ is defined as $cone(z) = \sum_{k=1}^r a_k \tau_k + \sum_{j=1}^q b_j Cone(\kappa_j)$, where $Cone(\kappa_j)$ is contained in $Cone\left(\partial_i\overline{M}\right)$. If two relative cycles z_1, z_2 are relatively homologous, then $cone(z_1)$ and $cone(z_2)$ represent the same homology class.

Since $M = \Gamma \backslash G/K$ is a finite-volume locally rank-one symmetric space, there is a homeomorphism $\Gamma \backslash G/K \cup \{\Gamma c_1, \ldots, \Gamma c_s\} \simeq DCone\left(\cup_{i=1}^s \partial_i\overline{M} \to \overline{M}\right)$, inducing the isomorphism $C_*\left(DCone\left(\cup_{i=1}^s \partial_i\overline{M} \to \overline{M}\right)\right) \simeq C_*(M \cup \{\Gamma c_1, \ldots, \Gamma c_s\})$.

Finally, str is the straightening which homotopes each cycle to a straight cycle with all interior vertices in x_0 and all ideal vertices remaining fixed during the homotopy. The same argument as in [1, C.4.3] (see the proof of Theorem 4 in [6]) shows that str exists and preserves homology classes.

Now, if z represents $\left[\overline{M}, \partial\overline{M}\right]$, then $w := str(cone(z))$ is an ideal fundamental cycle and, by Lemma 5, $\hat{\Psi}(w)$ is an ideal fundamental cycle, thus $i_1\left(cr\left(\hat{\Psi}(w)\right)\right)$ represents $\beta\left(\overline{M}\right)$ by Definition 3. Hence (omitting the index of $ev_{SL(N,\mathbf{C}),\Gamma_1',\ldots,\Gamma_s',\rho_\infty(c_0)}$ for the sake of readability)

$$\beta_\rho\left(\overline{M}\right) = \rho_*\beta\left(\overline{M}\right) = \rho_* i_{2*}\left[cr\hat{\Psi}(w)\right] = ev_*(B\rho)_* i_{1*}\left[\hat{\Phi}(w)\right] =$$

$$ev_*(B\rho)_* i_{1*}\hat{\Phi}_*\left[\overline{M}, \partial\overline{M}\right] = ev_*(B\rho)_* i_{1*}\left[\overline{M}, \partial\overline{M}\right] = ev_*\overline{\gamma}_\rho\left(\overline{M}\right).$$

Because of $ev_{SL(N,\mathbf{C}),\Gamma_1',\ldots,\Gamma_s',\rho_\infty(c_0)}j_* = ev_{SL(N,\mathbf{C})}$ this implies $ev\overline{\gamma}_\rho\left(\overline{M}\right) = \beta_\rho\left(\overline{M}\right)$. QED

In [8], A. A. Suslin constructed a homomorphism $K_3(\mathbf{C}) \otimes \mathbf{Q} \to \mathcal{B}(\mathbf{C}) \otimes \mathbf{Q}$ which yields an isomorphism $K_3^{ind}(\mathbf{C}) \otimes \mathbf{Q} \to \mathcal{B}(\mathbf{C}) \otimes \mathbf{Q}$.

The following corollary is well known to the experts, at least in the closed case, but seems not to have appeared in written form so far.

Corollary 2. *If M is an orientable, connected, hyperbolic 3-manifold of finite volume, then Suslin's homomorphism*

$$K_3(\mathbf{C}) \otimes \mathbf{Q} \to \mathcal{B}(\mathbf{C}) \otimes \mathbf{Q}$$

maps the Goncharov invariant $\gamma(M)$ to the Neumann-Yang invariant $\beta(M) \otimes 1$.

Proof: By [7, Section 5] we have $\beta(M) \in \mathcal{B}(\mathbf{C}) \subset \mathcal{P}(\mathbf{C})$.

[8] constructs a homomorphism $H_3(SL(3,\mathbf{C});\mathbf{Z}) \to \mathcal{B}(\mathbf{C})$ such that restriction to the image of $H_3(SL(2,\mathbf{C});\mathbf{Z})$ gives our evaluation map $ev_{SL(2,\mathbf{C})}$ (cf. [8, Lemma 3.4]). Composition with the inverse of the stability isomorphism $H_3(SL(3,\mathbf{C});\mathbf{Z}) \to H_3(SL(\mathbf{C});\mathbf{Z})$ yields a homomorphism $H_3(SL(\mathbf{C});\mathbf{Z}) \to \mathcal{B}(\mathbf{C})$ whose restriction to the image of $H_3(SL(2,\mathbf{C});\mathbf{Z})$ gives $ev_{SL(2,\mathbf{C})}$, in particular by Theorem 1 it maps $\overline{\gamma}(M)$ to $\beta(M) \otimes 1$.

Further composition with the map from [8, Corollary 5.2] gives the homomorphism $K_3(\mathbf{C}) \to \mathcal{B}(\mathbf{C})$. By [8, Theorem 5.1] it induces an isomorphism $K_3(\mathbf{C})/\pi_3\left(BGM(\mathbf{C})^+\right) \to \mathcal{B}(\mathbf{C})$. Indeed, [8, Theorem 5.1] gives an isomorphism $K_3(F)/\pi_3\left(BGM(F)^+\right) \to \mathcal{B}(F)/2c$ for any field F, where $c \in \mathcal{B}(F)$ is an element of order 6 by [8, Lemma 1.4]. In particular $c = 0$ since $\mathcal{B}(\mathbf{C})$ is a $\mathbf{Q}$-vector space. It follows then from [8, Theorem 5.2] that this morphism induces the isomorphism $K_3^{ind}(\mathbf{C}) \otimes \mathbf{Q} \to \mathcal{B}(\mathbf{C}) \otimes \mathbf{Q}$.

We note that the inclusion $K_3(\mathbf{C}) \otimes \mathbf{Q} \cong PH_3(GL(\mathbf{C});\mathbf{Q}) \cong PH_3(SL(\mathbf{C});\mathbf{Q}) \subset H_3(SL(\mathbf{C});\mathbf{Q})$ is actually an equality, hence (cf. [6, Section 2.5] and the definition of $\gamma(M)$ in [6, Theorem 4]) $\overline{\gamma}(M) = \gamma(M)$. And the claim follows. *QED*

References

[1] R. Benedetti, C. Petronio, 'Lectures on Hyperbolic Geometry', Universitext, Springer-Verlag, Berlin (1992).

[2] M. Bridson, A. Haefliger, 'Metric spaces of non-positive curvature', Grundlehren der Mathematischen Wissenschaften 319, Springer-Verlag, Berlin (1999).

[3] W. Dicks, M. J. Dunwoody, 'Groups acting on graphs'. Camb. Studies in Adv. Math. **17** (1989).

[4] P. Eberlein, 'Lattices in spaces of nonpositive curvature', *Ann. Math.* 111, pp.435-476 (1980).

[5] A. Goncharov, 'Volumes of hyperbolic manifolds and mixed Tate motives', *J. AMS* 12, pp.569-618 (1999).

[6] T. Kuessner, 'Locally symmetric spaces and K-theory of number fields', http://xxx.uni-augsburg.de/abs/0904.0919

[7] W. Neumann, J. Yang, 'Bloch invariants of hyperbolic 3-manifolds', *Duke Math. J.* 96, pp.29-59 (1999).

[8] A. A. Suslin, 'K_3 of a field, and the Bloch group', *Trudy Mat. Inst. Steklov* 183, pp.180-199 (1990).

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